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THE ASYMPTOTIC σ-ALGEBRA OF A RECURRENT RANDOM WALK ON A LOCALLY COMPACT GROUP

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ABSTRACT

Let μ be a probability measure on a locally compact second countable group G defining a recurrent (but not necessarily Harris) random walk. Denote by G^∞ the space of paths and by $\mathcal{B}^{(a)}$ the asymptotic σ-algebra. Let the starting measure be equivalent to the Haar measure and write Q for the corresponding Markov measure on G^∞. We prove that $L^\infty(G^\infty, \mathcal{B}^{(a)}, Q)$ is in a canonical way isomorphic to $L^\infty(G/N)$ where N is the smallest closed normal subgroup of G such that $\mu(zN) = 1$ for some $z \in G$. The group G/N is either a finite cyclic group with generator zN or a compact abelian group having the cyclic group $\{z^n N\}_{n\in\mathbb{Z}}$ as a dense subgroup. As a corollary we obtain that the set of all $\varphi \in L^1(G)$ such that $\lim_{n\to\infty} \|\varphi * \mu^n\|_1 = 0$ coincides with the kernel of the canonical mapping of $L^1(G)$ onto $L^1(G/N)$. In particular, when μ is aperiodic, i.e., $G = N$, then the random walk is mixing: $\lim_{n\to\infty} \|\varphi * \mu^n\|_1 = 0$ for every $\varphi \in L^1(G)$ with $\int \varphi = 0$.

1. Introduction

We shall consider right random walks on a locally compact second countable group G. A **right random walk** on G is a Markov chain with state space G and transition probability $\Pi(g, A) = \mu(g^{-1}A)$ where μ is a probability measure on G. The position of the random walk at time n is a product $X_n = Y_0 Y_1 \cdots Y_n$ where Y_0 is the initial position and $Y_1, Y_2, \ldots$ are independent G-valued random variables distributed according to the law μ. Y_0 is also a random variable, independent of $Y_1, Y_2, \ldots$ and distributed according to a law ν. We shall denote by $G^\infty = \prod_{n=0}^\infty G$

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the space of paths and by Q_ν the Markov measure on G^∞ determined by μ and ν. In the case that ν is a point measure δ_x, we shall write Q_x rather than Q_{δ_x}. $\mathcal{B}^{(a)}$ and $\mathcal{B}^{(i)}$ will stand for the asymptotic (tail) and invariant σ-algebras, respectively. The random walk will be called **adapted** when there is no proper closed subgroup $H \subseteq G$ with $\mu(H) = 1$, **irreducible** when there is no proper closed semigroup $S \subseteq G$ with $\mu(S) = 1$, and **aperiodic** when it is adapted and there is no proper closed normal subgroup $N \subseteq G$ and no $z \in G$ such that $\mu(zN) = 1$.

For any Borel subset $A \subseteq G$ let

$$r(A) = \{\{\omega_n\}_{n=1}^{\infty} \in G^\infty; \quad \omega_n \in A \text{ for infinitely many } n\text{'s}\}.$$

The set A is called **recurrent** if $Q_x(r(A)) = 1$ for all $x \in G$ and **transient** if $Q_x(r(A)) = 0$ for all $x \in G$. The random walk is called **recurrent** if every nonempty open set is recurrent and **transient** if every compact set is transient. An adapted random walk is either transient or recurrent [19, Chapter 3.3]. Every recurrent random walk is irreducible.

A random walk is called **recurrent in the sense of Harris** (or a **Harris random walk**) if every Borel set of nonzero Haar measure is recurrent. Recurrence in the sense of Harris implies recurrence but not every recurrent random walk is Harris. Call a random walk of law μ **spread out** if there exists $n = 1, 2, \ldots$ such that the convolution power μ^n is nonsingular with respect to the Haar measure. A random walk is recurrent in the sense of Harris if and only if it is recurrent and spread out [19, Theorem 3.9, p. 102].

The theory of Harris random walks is a special case of the well known theory of Markov chains recurrent in the sense of Harris [19]. The invariant σ-algebra of a Harris random walk is trivial in the following sense [19, Proposition 2.10, p. 94].

THEOREM 1.1: *If the random walk is Harris, then for every $A \in \mathcal{B}^{(i)}$ either $Q_x(A) = 0$ for all $x \in G$ or $Q_x(A) = 1$ for all $x \in G$.*

The asymptotic σ-algebra of a Harris random walk has a simple description in terms of a cyclic behavior of the random walk [19, Chapter 6.2]. Let N be the smallest closed normal subgroup of G such that $\mu(zN) = 1$ for some $z \in G$. It turns out that N is an open subgroup and G/N a finite cyclic group with generator zN. At each step the random walk proceeds from a coset z^nN to

$z^{n+1}N$ and returns to every coset periodically with period d, where d is the order of G/N. The asymptotic σ-algebra $\mathcal{B}^{(a)}$ is completely described in terms of sets $A_\xi \in \mathcal{B}^{(a)}$, $\xi \in G/N$, where

$$(1.1) \qquad A_\xi = \{\{\omega_n\}_{n=0}^{\infty} \in G^\infty;\ \text{for some } k, \quad \omega_n \in \xi(zN)^n \text{ for all } n \geq k\}.$$

Clearly, the A_ξ's are pairwise disjoint and $Q_x(A_\xi)$ is 1 for $x \in \xi$ and 0 otherwise.

THEOREM 1.2: *If the random walk is Harris, then for every $x \in G$ and every $A \in \mathcal{B}^{(a)}$, $Q_x(A)$ is either 0 or 1. Moreover, for every $A \in \mathcal{B}^{(a)}$ there exists a subset $I \subseteq G/N$ such that $Q_x(A \Delta \bigcup_{\xi \in I} A_\xi) = 0$ for all $x \in G$ where Δ denotes the symmetric difference.*

Theorems 1.1 and 1.2 fail for random walks that are recurrent but not Harris (i.e., not spread out). In fact, Theorem 1.1 is false for any random walk (recurrent or not) which is not spread out [19, Exercise 3.19, p. 105]. Furthermore, one can produce simple examples of recurrent random walks which admit invariant sets A such that $0 < Q_x(A) < 1$ for some $x \in G$ (see Example 4.1 below). In particular, both statements of Theorem 1.2 can be violated (note that the 2nd statement implies the 1st).

Bounded continuous harmonic functions of a recurrent random walk can be easily seen to be constant. One can use this to obtain the following substitute for Theorem 1.1 (see also [19, Exercises 4.13, p. 145 and 3.13, p. 59]).

THEOREM 1.3: *If the random walk is recurrent, then for every $A \in \mathcal{B}^{(i)}$ either $Q_x(A) = 1$ for λ-a.e. $x \in G$ or $Q_x(A) = 0$ for λ-a.e. $x \in G$, where λ is a Haar measure of G.*

When G/N is countable, N is necessarily open (by the Baire category theorem). Therefore, when the random walk is recurrent then G/N cannot be infinite countable. However, it can be uncountable, e.g., when G is the circle group and $\mu = \delta_g$ with $g = \exp(2\pi i r)$ and r irrational, or when $G = \mathbb{R}$ and μ has zero first order moment and is carried on two points generating a dense subgroup.

For every $\xi \in G/N$ define a set $A_\xi \in \mathcal{B}^{(a)}$ by

$$(1.2) \qquad A_\xi = \left\{\{\omega_n\}_{n=0}^{\infty} \in G^\infty;\ \lim_{n\to\infty} \omega_n z^{-n} N = \xi\right\}.$$

Note that when G/N is finite, G/N is discrete and then Formula (1.2) is equivalent to (1.1). We will prove the following substitute for Theorem 1.2.

THEOREM 1.4: *If the random walk is recurrent, then for every $A \in \mathcal{B}^{(a)}$, $Q_x(A) \in \{0,1\}$ for λ-a.e. $x \in G$. The group G/N is either a finite cyclic group or an uncountable compact abelian group having the cyclic group $\{z^n N\}_{n\in\mathbb{Z}}$ as a dense subgroup. For every $A \in \mathcal{B}^{(a)}$ there exists a Borel subset $I \subseteq G/N$ such that $\bigcup_{\xi\in I} A_\xi \in \mathcal{B}^{(a)}$ and $Q_x(A\Delta \bigcup_{\xi\in I} A_\xi) = 0$ for λ-a.e. $x \in G$.*

It is well known that the structure of the asymptotic σ-algebra is related to the asymptotic behavior of the convolution powers μ^n [3, 13]. Throughout the sequel we shall identify $L^1(G)$ with the space of complex measures absolutely continuous with respect to λ. Then $\|\cdot\|_1 = \|\cdot\| =$ total variation norm. We shall denote by $L^1_0(G) \subseteq L^1(G)$ the subspace of all $\varphi \in L^1(G)$ with $\varphi(G) = 0$. Recall that the random walk is called **ergodic** if

$$\lim_{n\to\infty} \left\| \frac{1}{n} \sum_{i=1}^{n} \varphi * \mu^i \right\| = 0$$

for all $\varphi \in L^1_0(G)$, and **mixing** if

$$\lim_{n\to\infty} \|\varphi * \mu^n\| = 0$$

for all $\varphi \in L^1_0(G)$. Ergodicity (resp., mixing) is equivalent to the condition that the space $L^\infty(G^\infty, \mathcal{B}^{(i)}, Q_\lambda)$ (resp., $L^\infty(G^\infty, \mathcal{B}^{(a)}, Q_\lambda)$) consists of constants only (see, e.g., [3, Théorème 6]). It is well known that every recurrent random walk is ergodic (an immediate consequence of Theorem 1.3). The fact that an aperiodic recurrent random walk is mixing (well known in the Harris case [19, Proposition 2.4, p. 196]) was recently proven by Lin and Wittmann in their study of convergence of averages of representations of G [14, Corollary 2.3]. Mixing follows immediately from our Theorem 1.4 and is a special case of the following more general result which we obtain along with Theorem 1.4. With π denoting the canonical homomorphism $\pi\colon G \to G/N$ we define a subspace $L^1_0(G,N) \subseteq L^1(G)$ by

$$L^1_0(G,N) = \left\{\varphi \in L^1(G);\quad \varphi(\pi^{-1}(A)) = 0 \text{ for every Borel subset } A \subseteq G/N\right\}.$$

Thus $L^1_0(G,N)$ is the kernel of the canonical mapping of $L^1(G)$ onto $L^1(G/N)$ and for $N = G$, $L^1_0(G,G) = L^1_0(G)$.

THEOREM 1.5: *If the random walk is recurrent, then for every $\varphi \in L^1(G)$*

$$\lim_{n\to\infty} \|\varphi * \mu^n\| = 0 \quad \textit{if and only if } \varphi \in L_0^1(G,N).$$

As shown in Section 4, the description of the asymptotic σ-algebra given in Theorem 1.4, as well as Theorem 1.5, remain valid not only for recurrent random walks but, more generally, for random walks obeying the following weak 0–1 law: for every $A \in \mathcal{B}^{(a)}$, $Q_x(A) \in \{0,1\}$ for λ-a.e. $x \in G$. This includes arbitrary random walks on abelian G. In the abelian case the set

$$\{\varphi \in L^1(G);\ \lim_{n\to\infty} \|\varphi * \mu^n\| = 0\} = L_0^1(G,N)$$

can be also charactersized in terms of the dual group and the Fourier transforms of μ and φ [7, 8].

2. Random walks and harmonic functions

Let G be a locally compact second countable (lcsc) group. We shall denote by G^∞ the product space $G^\infty = \prod_{n=0}^\infty G$, by $X_n\colon G^\infty \to G$, $n = 0,1,\ldots$, the canonical projections, and by $\mathcal{B}^\infty$ the product σ-algebra $\mathcal{B}^\infty = \prod_{n=0}^\infty \mathcal{B} = \sigma(X_0, X_1, \ldots)$, where $\mathcal{B}$ is the σ-algebra of Borel subsets of G. λ will stand for a Haar measure.

Let μ be a probability measure on G and write Q_g for the Markov measure of the random walk (of law μ) started from a point $g \in G$. Given $A \in \mathcal{B}^\infty$, the function $G \ni g \to Q_g(A)$ is Borel. When ν is a measure on G, the Markov measure Q_ν of the random walk whose starting measure is ν can be expressed as

$$Q_\nu(A) = \int_G \nu(dg) Q_g(A), \quad A \in \mathcal{B}^\infty. \tag{2.1}$$

It is clear that the measure class of Q_ν is completely determined by the measure class of ν, i.e., if $\nu \sim \nu'$ then $Q_\nu \sim Q_{\nu'}$.

The transition probability $\Pi(g,A) = \mu(g^{-1}A)$ is invariant with respect to the action of G on G given by multiplication on the left, i.e., $P(gg', gA) = P(g', A)$ for all $g, g' \in G$ and $A \in \mathcal{B}$. There is also a natural action of G on the space of paths G^∞, namely, $g\{\omega_n\}_{n=1}^\infty = \{g\omega_n\}_{n=0}^\infty$. $(G^\infty, \mathcal{B}^\infty)$ is a Borel G-space. The Markov measures Q_g satisfy $Q_{gg'}(gA) = Q_{g'}(A)$, $g, g' \in G$, $A \in \mathcal{B}^\infty$. Hence, when ν is equivalent to the Haar measure λ then Q_ν is a quasiinvariant measure on the

G-space $(G^\infty, \mathcal{B}^\infty)$. We have a natural G-action on $L^\infty(G^\infty, \mathcal{B}^\infty, Q_\lambda)$ given by $(gf)(\omega) = f(g^{-1}\omega)(\mathrm{mod}\, Q_\lambda)$, $g \in G$, $f \in L^\infty(G^\infty, \mathcal{B}^\infty, Q_\lambda)$. The formula

$$(2.2) \qquad (Rf)(g) = \int_{G^\infty} Q_g(d\omega) f(\omega) \quad (\mathrm{mod}\,\lambda)$$

defines an equivariant contraction R: $L^\infty(G^\infty, \mathcal{B}^\infty, Q_\lambda) \to L^\infty(G)$.

Let ϑ: $G^\infty \to G^\infty$ denote the shift $\vartheta\{\omega_n\}_{n=0}^\infty = \{\omega_{n+1}\}_{n=0}^\infty$. ϑ transforms the Markov measure Q_ν into the Markov measure $Q_{\nu*\mu}$. I.e.,

$$(2.3) \qquad (\vartheta Q_\nu)(A) = Q_\nu(\vartheta^{-1}(A)) = Q_{\nu*\mu}(A), \quad A \in \mathcal{B}^\infty.$$

When $\nu \sim \lambda$ then $\vartheta Q_\nu \sim Q_\nu$ and, hence, the formula $\theta f = f \circ \vartheta$ $(\mathrm{mod}\, Q_\lambda)$, $f \in L^\infty(G^\infty, \mathcal{B}^\infty, Q_\lambda)$, defines an injective homomorphism θ of the $*$-algebra $L^\infty(G^\infty, \mathcal{B}^\infty, Q_\lambda)$ into itself. It is clear that θ and the G-action commute. We also note that

$$(2.4) \qquad R\theta = PR$$

where P: $L^\infty(G) \to L^\infty(G)$ is the contraction induced by Π (i.e., by μ),

$$(2.5) \qquad (Pf)(g) = \int_G \Pi(g, dg') f(g') \quad (\mathrm{mod}\,\lambda).$$

The **asymptotic σ-algebra $\mathcal{B}^{(a)}$**,

$$(2.6) \qquad \mathcal{B}^{(a)} = \bigcap_{k=0}^{\infty} \sigma(X_k, X_{k+1}, \ldots),$$

is invariant under the G-action. Consequently, $L^\infty(G^\infty, \mathcal{B}^{(a)}, Q_\lambda)$ is a G-invariant subalgebra of $L^\infty(G^\infty, \mathcal{B}^\infty, Q_\lambda)$. We shall use the notation $L_a^\infty(\mu)$ instead of the cumbersome $L^\infty(G^\infty, \mathcal{B}^{(a)}, Q_\lambda)$.

The map $\mathcal{B}^{(a)} \ni A \to \vartheta^{-1}(A) \in \mathcal{B}^{(a)}$ is an automorphism of $\mathcal{B}^{(a)}$. Hence, the homomorphism θ restricted to $L_a^\infty(\mu)$ is an automorphism. In the sequel we shall consider $L_a^\infty(\mu)$ as a $(G \times \mathbb{Z})$-space with the $(G \times \mathbb{Z})$-action given by $(g,k)f = g\theta^k f$, $g \in G$, $k \in \mathbb{Z}$, $f \in L_a^\infty(\mu)$.

Let $\mathcal{H}^\infty$ denote the space of all sequences $h = \{h_n\}_{n=0}^\infty$ where $h_n \in L^\infty(G)$, $\sup_n \|h_n\| < \infty$, and $h_n = Ph_{n+1}$ for all $n = 0, 1, \ldots$. $\mathcal{H}^\infty$ equipped with the norm $\|h\| = \sup_n \|h_n\|$ is a Banach space. It is also a G-space with the G-action

$gh = g\{h_n\}_{n=0}^{\infty} = \{gh_n\}_{n=0}^{\infty}$. Elements of $\mathcal{H}^\infty$ will be referred to as **space-time harmonic functions**.

Let $f \in L_a^\infty(\mu)$ and $h_n = R\theta^{-n}f$. From (2.4) we have that $\{h_n\}_{n=0}^{\infty}$ is a space-time harmonic function. The following basic result is a well known consequence of the martingale convergence theorem [17].

THEOREM 2.1: *The map $\mathcal{R}$: $L_a^\infty(\mu) \to \mathcal{H}^\infty$ defined by $\mathcal{R}f = \{R\theta^{-n}f\}_{n=0}^{\infty}$ is an equivariant isometric isomorphism of $L_a^\infty(\mu)$ onto $\mathcal{H}^\infty$. Moreover, for every $h = \{h_n\}_{n=0}^{\infty} \in \mathcal{H}^\infty$ the sequence $\{h_n \circ X_n\}_{n=0}^{\infty}$ converges Q_λ-a.e. to $\mathcal{R}^{-1}h$.*

The σ-algebra $\mathcal{B}^{(i)} = \{A \in \mathcal{B}^\infty;\ \vartheta^{-1}(A) = A\}$ is called the **invariant σ-algebra**. We shall write $L_i^\infty(\mu)$ for $L^\infty(G^\infty, \mathcal{B}^{(i)}, Q_\lambda)$. $L_i^\infty(\mu)$ is the fixed point algebra of the homomorphism θ: $L^\infty(G^\infty, \mathcal{B}^\infty, Q_\lambda) \to L^\infty(G^\infty, \mathcal{B}^\infty, Q_\lambda)$ and is G-invariant since θ and the G-action commute.

As follows from (2.4), when $f \in L_i^\infty(\mu)$ then $PRf = Rf$. Fixed points of the contraction P will be called **harmonic functions** and $\mathcal{H} \subseteq L^\infty(G)$ will denote the space of harmonic functions.

THEOREM 2.2: *R (restricted to $L_i^\infty(\mu)$) is an equivariant isometric isomorphism of $L_i^\infty(\mu)$ onto $\mathcal{H}$. Moreover, for every $h \in \mathcal{H}$ the sequence $\{h \circ X_n\}_{n=0}^{\infty}$ converges Q_λ-a.e. to $R^{-1}h$.*

This theorem, similarly as Theorem 2.1, is a consequence of the martingale convergence theorem [17]. We remark that the space-time harmonic functions are often introduced as the harmonic functions of the space-time process, i.e., the random walk of law $\mu \times \delta_1$ on $G \times \mathbb{Z}$ [19]. The space $L_a^\infty(\mu)$ is canonically isomorphic to $L_i^\infty(\mu \times \delta_1)$.

We end this section by quoting an important result of Derriennic [3, Théorème 1]. When φ is a complex measure, we denote by $\|\varphi\|$ the total variation norm. When φ is a complex measure on a σ-algebra containing $\mathcal{B}^{(a)}$ (resp., $\mathcal{B}^{(i)}$), we write $\|\nu\|_a$ (resp., $\|\nu\|_i$) for the total variation norm of the restriction of φ to $\mathcal{B}^{(a)}$ (resp., $\mathcal{B}^{(i)}$).

THEOREM 2.3: *When φ is a complex measure on G then*

$$\lim_{n\to\infty} \|\varphi * \mu^n\| = \|Q_\varphi\|_a, \qquad \lim_{n\to\infty} \left\| \frac{1}{n} \sum_{i=1}^{n} \varphi * \mu^i \right\| = \|Q_\varphi\|_i.$$

3. Boundaries

By a G-space $(\mathcal{X}, \mathcal{A}, \alpha)$ we mean a Borel G-space $(\mathcal{X}, \mathcal{A})$ with a quasiinvariant measure α. When $(\mathcal{X}, \mathcal{A}, \alpha)$ and $(\mathcal{X}', \mathcal{A}', \alpha')$ are two such G-spaces we say that $L^\infty(\mathcal{X}, \mathcal{A}, \alpha)$ is G-isomorphic to $L^\infty(\mathcal{X}', \mathcal{A}', \alpha')$ if there exists an equivariant isomorphism of the $*$-algebra $L^\infty(\mathcal{X}, \mathcal{A}, \alpha)$ onto $L^\infty(\mathcal{X}', \mathcal{A}', \alpha')$. We define the **$\mu$-boundary** of a random walk of law μ as any G-space $(\mathcal{X}, \mathcal{A}, \alpha)$ such that $L^\infty(\mathcal{X}, \mathcal{A}, \alpha)$ is G-isomorphic to $L_i^\infty(\mu)$. We define the **space-time μ-boundary** as any $(G \times \mathbb{Z})$-space $(\mathcal{X}, \mathcal{A}, \alpha)$ such that $L^\infty(\mathcal{X}, \mathcal{A}, \alpha)$ is $(G \times \mathbb{Z})$-isomorphic to $L_a^\infty(\mu)$. We remark that since Q_λ is equivalent to a finite measure, the same is true for the quasiinvariant measure α on any boundary or space-time boundary.

A G-space $(\mathcal{X}, \mathcal{A}, \alpha)$ is a μ-boundary if and only if there exists an equivariant identity preserving isometry U of $L^\infty(\mathcal{X}, \mathcal{A}, \alpha)$ onto the space $\mathcal{H}$ of harmonic functions. A $(G \times \mathbb{Z})$-space $(\mathcal{X}, \mathcal{A}, \alpha)$ is a space-time μ-boundary if and only if there exists an identity preserving contraction $V\colon L^\infty(\mathcal{X}, \mathcal{A}, \alpha) \to L^\infty(G)$ such that the map $L^\infty(\mathcal{X}, \mathcal{A}, \alpha) \ni f \to \{V(e, -n)f\}_{n=0}^\infty$ is an isometry onto the space $\mathcal{H}^\infty$ of space-time harmonic functions (e is the identity of G) and $V(g, 0)f = gVf$ for $g \in G$ and $f \in L^\infty(\mathcal{X}, \mathcal{A}, \alpha)$. (The "if" part of this characterization relies on the fact that an identity preserving surjective isometry between two abelian C^*-algebras is a $*$-isomorphism [2, Corollaire on p. 7].)

Clearly, $(G^\infty, \mathcal{B}^{(i)}, Q_\lambda)$ is a boundary. $(G^\infty, \mathcal{B}^{(a)}, Q_\lambda)$ is not exactly a space-time boundary because the shift ϑ is not invertible on G^∞. In any case the spaces $(G^\infty, \mathcal{B}^{(i)}, Q_\lambda)$ and $(G^\infty, \mathcal{B}^{(a)}, Q_\lambda)$ are awkward to work with, one reason being that the σ-algebras $\mathcal{B}^{(i)}$ and $\mathcal{B}^{(a)}$ do not separate points. However, the fact that G is lcsc allows a routine application of the classical Mackey theorem about pointwise realizations of group actions [15]. This shows that there always exist boundaries $(\mathcal{X}, \mathcal{A}, \alpha)$ (resp., space-time boundaries) that are **standard**, i.e., standard in their Borel structure and such that the map $G \times \mathcal{X} \ni (g, x) \to gx$ (resp., $(G \times \mathbb{Z}) \times \mathcal{X} \ni ((g, k), x) \to (g, k)x$) is Borel. Furthermore, a theorem of Varadarajan [21, Theorem 3.2] shows that one can even assume $\mathcal{X}$ to be compact metric and the map $G \times \mathcal{X} \ni (g, x) \to gx$ (resp., $(G \times \mathbb{Z}) \times \mathcal{X} \ni ((g, k), x) \to (g, k)x$) to be continuous. Below, by a continuous boundary (resp., space-time boundary) we mean a boundary (resp., space-time boundary) $(\mathcal{X}, \mathcal{A}, \alpha)$ such that $\mathcal{X}$ is a lcsc (Hausdorff) space and the G-action (resp., $(G \times \mathbb{Z})$-action) is continuous as above.

When α is a σ-finite measure on a Borel space $(\mathcal{X}, \mathcal{A})$ we shall identify $L^1(\mathcal{X}, \mathcal{A}, \alpha)$ with the space of complex measures absolutely continuous with respect to α. Then $\|\cdot\|_1 = \|\cdot\| =$ the total variation norm. When $(\mathcal{X}, \mathcal{A})$ and $(\mathcal{X}', \mathcal{A}')$ are Borel spaces, $F\colon \mathcal{X} \to \mathcal{X}'$, a Borel map and ν a measure on $\mathcal{A}$, we shall write $F\nu$ for the measure $(F\nu)(A) = \nu(F^{-1}(A))$, $A \in \mathcal{A}'$. In particular, when $(\mathcal{X}, \mathcal{A})$ is a Borel G-space, $g\nu$ is the measure $(g\nu)(A) = \nu(g^{-1}A)$.

Given a space-time μ-boundary $(\mathcal{X}, \mathcal{A}, \alpha)$ we shall denote by $\hat{\vartheta}$ the automorphism $\hat{\vartheta}x = (e, -1)x$. We shall write $g\hat{\vartheta}^{-k}x$ rather than $(g, k)x$, $g \in G$, $k \in \mathbb{Z}$, $x \in \mathcal{X}$. When μ is a measure on G and ρ a measure on $\mathcal{X}$, the convolution $\mu * \rho$ is defined by

$$(\mu * \rho)(A) = \int_G \mu(dg)\rho(g^{-1}A), \quad A \in \mathcal{A}.$$

When $\mathcal{X}$ is a lcsc space, by the weak$*$ topology on the space $\mathcal{M}$ of complex measures on $\mathcal{X}$ we mean the $\sigma(\mathcal{M}, C_0(\mathcal{X}))$-topology, where $C_0(\mathcal{X})$ is the space of continuous functions vanishing at infinity.

The following proposition establishes basic properties of the continuous space-time μ-boundary. The proof is technical and we relegate it to the Appendix.

PROPOSITION 3.1: *Let μ be a probability measure on G and $(\mathcal{X}, \mathcal{A}, \alpha)$ a continuous space-time μ-boundary. Let Φ denote an equivariant isomorphism of $L^\infty(\mathcal{X}, \mathcal{A}, \alpha)$ onto $L^\infty_a(\mu)$. It follows that there exists a probability measure ρ on $\mathcal{X}$ such that*

(a) $\mu * \rho = \hat{\vartheta}\rho$,
(b) $\lambda * \hat{\vartheta}^n\rho \sim \alpha$ *for all* $n \in \mathbb{Z}$,
(c) $(R\theta^{-n}\Phi f)(g) = \int_{\mathcal{X}} (\hat{\vartheta}^{-n}\rho)(dx) f(gx) \pmod{\lambda}$ *for all* $f \in L^\infty(\mathcal{X}, \mathcal{A}, \alpha)$ *and* $n = 0, 1, \ldots$.
(d) $\lim_{n\to\infty} \|\varphi * \mu^n\| = \|\varphi * \rho\|$ *for every* $\varphi \in L^1(G)$.

Moreover, if $\Omega = \{\omega = \{\omega_n\}_{n=0}^\infty \in G^\infty$; the sequence $\omega_n\hat{\vartheta}^{-n}\rho$ converges weakly$*$ to a point measure$\}$, then $\Omega \in \mathcal{B}^{(a)}$ and $Q_\lambda(G^\infty - \Omega) = 0$. There exists a Borel map $F\colon G^\infty \to \mathcal{X}$ such that $FQ_\lambda \sim \alpha$, $\Phi f = f \circ F$ for every $f \in L^\infty(\mathcal{X}, \mathcal{A}, \alpha)$, and $\delta_{F(\omega)} = \lim_{n\to\infty} \omega_n\hat{\vartheta}^{-n}\rho$ for every $\omega \in \Omega$. Furthermore, if $\Omega_1 \in \mathcal{B}^{(a)}$ and $Q_\lambda(G^\infty - \Omega_1) = 0$ then for α-a.e. $x \in \mathcal{X}$ there is $\omega \in \Omega_1$ with $\omega_n\hat{\vartheta}^{-n}\rho \to \delta_x$ weakly$*$.

The Proposition has an obvious analog for the continuous μ-boundary. (Formally, one puts $\hat{\vartheta} =$ id and replaces $\|\varphi * \mu^n\|$ by $\|(1/n)\sum_{i=1}^n \varphi * \mu^i\|$ in statement (d).)

Our definitions of the continuous μ-boundary and space-time μ-boundary, as well as Proposition 3.1, are directly motivated by the work of Furstenberg [9, 10] to whom the concept of a continuous μ-boundary is due. Our continuous μ-boundary corresponds to Furstenberg's 'universal μ-boundary' [10] or 'Poisson boundary' [9], although his definition differs from ours and contains some of the statements of the analog of Proposition 3.1 as defining properties. The continuous space-time μ-boundary will be indispensable for our proof of Theorems 1.4 and 1.5.

4. Weak 0–1 law and the boundaries

We shall say that a random walk on G obeys the 0–1 law if for every $x \in G$ and every $A \in \mathcal{B}^{(a)}$, $Q_x(A)$ is either 0 or 1. By Theorem 1.2 a Harris random walk obeys the 0–1 law. When G is abelian, every random walk obeys the 0–1 law as a consequence of the Hewitt–Savage 0–1 law [12], [16, Chapter VIII.I]. However, not every recurrent random walk obeys the 0–1 law:

Example 4.1: Consider any nonabelian compact connected Lie group G (e.g, $\mathrm{SO}(3,\mathbb{R})$). It is well known that considered as a discrete group, G is not amenable and contains a free group on two generators as a subgroup [18, Theorem 3.9, p. 107 and B51, p. 324]. Let D be a countable dense subgroup of G containing a free group and let μ be a probability measure carried on D and such that the support of μ generates D. Then μ is adapted and as G is compact μ defines a recurrent random walk on G. Suppose that the 0–1 law holds. Then from (2.1) and (2.3) with $\nu = \delta_e$ we obtain $Q_x(A) = Q_e(A)$ for all $A \in \mathcal{B}^{(i)}$ and all $x \in \operatorname{supp}\mu$. As $\operatorname{supp}\mu$ generates D, $Q_x(A) = Q_e(A)$ for all $A \in \mathcal{B}^{(i)}$ and $x \in D$. Using Theorem 2.3 we then have

$$\lim_{n\to\infty}\left\| x\left(\frac{1}{n}\sum_{i=1}^{n}\mu^i\right) - \frac{1}{n}\sum_{i=1}^{n}\mu^i\right\| = \lim_{n\to\infty}\left\|\frac{1}{n}\sum_{i=1}^{n}(\delta_x-\delta_e)*\mu^i\right\| = 0$$

for all $x \in D$. As μ is carried on D, this would imply that D is amenable [11, Theorem 2.4.3], a contradiction since D contains a free group.

We shall say that a random walk obeys the **weak 0–1 law** if for every $A \in \mathcal{B}^{(a)}$ $Q_x(A) \in \{0,1\}$ for λ-a.e. $x \in G$. We remark that for a spread out random walk the 0–1 law and the weak 0–1 law are equivalent. This follows from the fact that bounded space-time harmonic functions of a spread out random walk are continuous (apply [19, Proposition 1.6, p. 162] to $(\mu\times\delta_1)$-harmonic functions).

PROPOSITION 4.2: *Let $(\mathcal{X}, \mathcal{A}, \alpha)$ be a continuous space-time μ-boundary. The random walk obeys the weak 0–1 law if and only if the measure ρ of Proposition 3.1 is a point measure.*

Proof: We leave it to the reader to check that the weak 0–1 law is equivalent to the condition that the contraction R of Equation (2.2) be multiplicative on $L^\infty_a(\mu)$. Hence, with the notation of Proposition 3.1, the weak 0–1 law is equivalent to the condition that $R\Phi$ be multiplicative. But $(R\Phi f)(g) = \int \rho(dx)f(gx)$ $(\operatorname{mod}\lambda)$ for all $f \in L^\infty(\mathcal{X}, \mathcal{A}, \alpha)$. Clearly, if ρ is a point measure, then $R\Phi$ is multiplicative. To establish the converse, note that for $f \in C_0(\mathcal{X})$ the function $G \ni g \to \int \rho(dx)f(gx)$ is continuous. Hence, if $R\Phi$ is multiplicative, then the functional $C_0(\mathcal{X}) \ni f \to \int \rho(dx)f(x)$ is multiplicative. Thus ρ must be a point measure. ∎

Given a probability measure μ on G we shall denote by H the smallest closed subgroup with $\mu(H) = 1$ and by N the smallest closed normal subgroup of H such that $\mu(zN) = 1$ for some $z \in H$.

PROPOSITION 4.3:

(a) *If H/N is countable, it is discrete, cyclic and generated by zN.*
(b) *If H/N is not countable, it is compact abelian and the cyclic group generated by zN is dense in H/N.*
(c) *If H coincides with the smallest closed semigroup S such that $\mu(S) = 1$, then H/N is compact.*

Proof: The Proposition (which is not difficult to prove) is essentially a reformulation of Proposition 1.6 in [6]. ∎

Note that G/N admits a homeomorphism $\hat{\vartheta}$: $G/N \to G/N$ such that $\hat{\vartheta}(gN) = gzN$ for every $g \in G$. $\hat{\vartheta}$ commutes with the usual G-action on G/N and preserves the (unique) G-invariant measure class of G/N. By the $(G \times \mathbb{Z})$-action on G/N we shall mean the action $(g, k)x = g\hat{\vartheta}^{-k}x$. Note that when $G = H$ then $\hat{\vartheta}$ is simply the translation $\hat{\vartheta}(gN) = (gN)(zN)$ on the abelian group G/N; $\hat{\vartheta}$ acts either transitively (case (a) of Proposition 4.3) or properly ergodically (case (b)).

For every $n = 0, 1, \dots$ define a map Ψ_n: $L^\infty(G/N) \to L^\infty(G)$ by $(\Psi_n f)(x) = f(xz^{-n}N) = f(\hat{\vartheta}^{-n}(xN))$ $(\operatorname{mod}\lambda)$. It is clear that given $f \in L^\infty(G/N)$, the sequence $\{\Psi_n f\}_{n=0}^\infty$ is a space-time harmonic function. Therefore one can define

$\Phi\colon L^{\infty}(G/N) \to L^{\infty}_a(\mu)$ by

$$\Phi f = \mathcal{R}^{-1}\{\Psi_n f\}_{n=0}^{\infty} \tag{4.1}$$

where $\mathcal{R}$ is the isomorphism of Theorem 2.1. It easily follows that Φ is a $(G\times\mathbb{Z})$-equivariant isometry of $L^{\infty}(G/N)$ *into* $L^{\infty}_a(\mu)$.

We shall say that $g \in G$ is a period of $\mathcal{H}^{\infty}$ if for every $h = \{h_n\}_{n=0}^{\infty} \in \mathcal{H}^{\infty}$ we have $h_n(xg) = h_n(x) \pmod{\lambda}$ for every $n = 0, 1, \ldots$. Of course, the periods form a subgroup of G.

THEOREM 4.4: *The following conditions are equivalent for a random walk of law μ:*

(a) *the random walk obeys the weak 0–1 law;*
(b) *N is contained in the group of periods of $\mathcal{H}^{\infty}$;*
(c) *the mapping Φ of Equation (4.1) is an equivariant isomorphism of $L^{\infty}(G/N)$ onto $L^{\infty}_a(\mu)$;*
(d) *G/N is a space-time μ-boundary and the probability measure ρ of Proposition 3.1 can be chosen as $\rho = \delta_N$;*
(e) $\lim_{n\to\infty} \|\varphi * \mu^n\| = 0$ *for every $\varphi \in L^1_0(G,N)$.*

Proof: (a)⇒(b): Let $(\mathcal{X}, \mathcal{A}, \alpha)$ be some continuous space-time μ-boundary and ρ the probability measure of Proposition 3.1. Then ρ is a point measure δ_p and by Proposition 3.1(a) $gp = \hat{\vartheta}p$ for every $g \in \operatorname{supp}\mu$ (here $\hat{\vartheta}$ is as in Proposition 3.1). Since $\hat{\vartheta}$ commutes with the G action, we conclude that $gp = p$ for all $g \in \bigcup_{n=1}^{\infty}(S_1^n S_1^{-n} \cup S_1^{-n} S_1^n)$ where $S_1 = \operatorname{supp}\mu$. But N is the smallest closed subgroup containing $\bigcup_{n=1}^{\infty}(S_1^n S_1^{-n} \cup S_1^{-n} S_1^n)$ [6, Proposition 1.1]. Therefore $gp = p$ for $g \in N$. Using now point (c) of Proposition 3.1 we obtain that if $h = \{h_n\}_{n=0}^{\infty}$ is a space-time harmonic function, then, given $g \in N$, $h_n(xg) = h_n(x) \pmod{\lambda}$ for every n.

(b)⇒(c): It suffices to show that for every $h = \{h_n\}_{n=0}^{\infty} \in \mathcal{H}^{\infty}$ there exists $f \in L^{\infty}(G/N)$ such that $h_n(x) = f(xz^{-n}N) \pmod{\lambda}$ for all $n = 0, 1, \ldots$. But this follows immediately from property (b) and the definition of space-time harmonic functions.

(c)⇒(d): Obvious.

(d)⇒(a): Obvious by Proposition 4.2.

(d)⇒(e): Obvious by Proposition 3.1(d).

(e)⇒(b): Let $\nu_n \ll \lambda$ be a sequence of probability measures converging weakly$*$ to δ_e. Let $g \in N$. Then for every n, $\nu_n * \delta_g - \nu_n \in L^1_0(G,N)$. Hence, if $(\mathcal{X}, \mathcal{A}, \alpha)$ is some continuous space-time μ-boundary and ρ the probability measure of Proposition 3.1, then $(\nu_n * \delta_g) * \rho - \nu_n * \rho = \nu_n * g\rho - \nu_n * \rho = 0$ by Proposition 3.1(d). Passing to the limit $n \to \infty$ we obtain $g\rho = \rho$. Proposition 3.1(c) then implies (b). ∎

COROLLARY 4.5: *If the random walk obeys the weak 0–1 law, then*

$$N = \overline{\bigcup_{n=1}^{\infty} (\operatorname{supp} \mu)^n (\operatorname{supp} \mu)^{-n}}.$$

Proof: Let $S_n = \operatorname{supp} \mu^n = \overline{(\operatorname{supp} \mu)^n}$. It is clear that

$$\overline{\bigcup_{n=1}^{\infty} S_n S_n^{-1}} = \overline{\bigcup_{n=1}^{\infty} S_1^n S_1^{-n}} \subseteq N.$$

Suppose that $g \in N - \overline{\bigcup_{n=1}^{\infty} S_n S_n^{-1}}$. Then there is a neighbourhood U of e in G such that $Ug \cap \overline{\bigcup_{n=1}^{\infty} S_n S_n^{-1}} = \emptyset$. If V is a neighbourhood of e such that $V^{-1}V \subseteq U$, then $VgS_n \cap VS_n = \emptyset$ for all $n = 1, 2, \ldots$. Let ν be an absolutely continuous probability measure carried on V. Then $(\nu * \delta_g) * \mu^n$ is carried on VgS_n while $\nu * \mu^n$ is carried on VS_n. Hence, $\|(\nu * \delta_g) * \mu^n - \nu * \mu^n\| = 2$. But $\nu * \delta_g - \nu \in L_0^\infty(G, N)$ and we obtain a contradiction with Theorem 4.4(e). ∎

COROLLARY 4.6: *If the random walk obeys the weak 0–1 law, then for every $\varphi \in L^1(G)$, $\lim_{n\to\infty} \|\varphi * \mu^n\| = 0$ if and only if $\varphi \in L_0^1(G, N)$.*

Proof: Combine Theorems 3.1(d) and 4.4(d), (e). ∎

Let, for every $\xi \in G/N$, A_ξ denote the set

$$A_\xi = \left\{ \{\omega_n\}_{n=0}^{\infty} \in G^\infty;\ \lim_{n\to\infty} \omega_n z^{-n} N = \xi \right\}.$$

Since G/N is 2nd countable, it follows that $A_\xi \in \mathcal{B}^{(a)}$.

COROLLARY 4.7: *If the random walk obeys the weak 0–1 law, then for every $A \in \mathcal{B}^{(a)}$ there exists a Borel subset $I \subseteq G/N$ such that $\bigcup_{\xi \in I} A_\xi \in \mathcal{B}^{(a)}$ and $Q_\lambda(A \Delta \bigcup_{\xi\in I} A_\xi) = 0$.*

Proof: Apply the Borel map $F\colon G^\infty \to G/N$ of Proposition 3.1. ∎

Remarks: (1) It can be shown that the weak 0–1 law is equivalent to the condition that the random walk on $G \times \mathbb{Z}$ of law $\mu \times \delta_1$ satisfy the Choquet–Deny theorem: bounded continuous harmonic functions are constant on the left cosets of the smallest closed subgroup $\tilde{H} \subseteq G \times \mathbb{Z}$ with $(\mu \times \delta_1)(\tilde{H}) = 1$. In [5] Derriennic and Lin showed that under the latter condition G/N is a space-time μ-boundary.

(2) Since for a mixing random walk $L^\infty_a(\mu) = \mathbb{C}1\!\!1$, Theorem 4.4 and Corollary 4.5 contain a recent result of Lin and Wittmann [14, Corollary 2.7(i)]: the condition $G = \overline{\bigcup_{n=1}^\infty (\operatorname{supp}\mu)^n (\operatorname{supp}\mu)^{-n}}$ is necessary for mixing.

(3) Using one of Derriennic's 0–2 laws [3] one can show that the 0–1 law is equivalent to the following **restricted 0–1 law**: for every $x \in G$ and every $A \in \mathcal{B}^{(i)}$, $Q_x(A)$ is either 0 or 1 [4, Appendice 2]. Motivated by this result one may conjecture that the weak 0–1 law is equivalent to the **weak restricted 0–1 law**: for every $A \in \mathcal{B}^{(i)}$, $Q_x(A) \in \{0, 1\}$ for λ-a.e. $x \in G$. The weak restricted 0-1 law is equivalent to the Choquet–Deny theorem. Using the μ-boundary analog of Proposition 3.1 one can in an obvious way modify the proofs of Proposition 4.2, Theorem 4.4, and Corollaries 4.5–4.7 to obtain the corresponding results for the weak restricted 0–1 law. In particular, the result of Lin and Wittman [14, Corollary 2.7(ii)] follows: the condition $G = \overline{\left(\bigcup_{n=1}^\infty (\operatorname{supp}\mu)^n\right)\left(\bigcup_{n=1}^\infty (\operatorname{supp}\mu)^n\right)^{-1}}$ is necessary for ergodicity of the random walk.

5. Recurrent random walks

Using the μ-boundary analogs of Propositions 3.1 and 4.2 it is almost trivial to show that a recurrent random walk obeys the weak restricted 0–1 law. We will show that it also obeys the weak 0–1 law.

THEOREM 5.1: *Every recurrent random walk on a lcsc group obeys the weak 0–1 law.*

Proof: Let $(\mathcal{X}, \mathcal{A}, \alpha)$ be a continuous space-time μ-boundary. Note that since α is quasiinvariant, $\operatorname{supp}\alpha$ is a closed G-invariant set. Hence, replacing $\mathcal{X}$ by $\operatorname{supp}\alpha$ we can assume that $\alpha(U) \neq 0$ for every nonempty open subset of $\mathcal{X}$.

We shall use the notation of Proposition 3.1. We claim that there exists an α-conull Borel set $\mathcal{X}_0 \subseteq \mathcal{X}$ such that for every $x \in \mathcal{X}_0$ there is a sequence $\{n_k\}_{k=0}^\infty$ of positive integers such that $\hat{\vartheta}^{-n_k}\rho$ converges weakly$*$ to δ_x.

Let $\{U_k\}_{k=0}^\infty$ be a decreasing neighbourhood base at $e \in G$ and let $\Omega_1 = \bigcap_{k=0}^\infty r(U_k)$. Since the random walk is recurrent, $Q_\lambda(G^\infty - \Omega_1) = 0$. Using the last statement of Proposition 3.1 there exists a conull Borel set $\mathcal{X}_0 \subseteq \mathcal{X}$ such that for every $x \in \mathcal{X}_0$ there is $\omega = \{\omega_n\}_{n=0}^{\infty} \in \Omega_1$ with $\omega_n \hat{\vartheta}^{-n}\rho \to \delta_x$ weakly*. Since $\omega \in \Omega_1$, there is a subsequence $\{\omega_{n_k}\}_{k=0}^{\infty}$ with $\omega_{n_k} \in U_k$, i.e., $\lim_{k\to\infty} \omega_{n_k} = e$. Using the fact that for every $f \in C_0(\mathcal{X})$ the map $G \ni g \to gf$ is continuous with respect to the sup norm, we conclude that $\hat{\vartheta}^{-n_k}\rho \to \delta_x$ weakly*. Thus $\mathcal{X}_0$ has the desired property.

We now claim that N is contained in the stabilizer of every point $x \in \mathcal{X}_0$. Indeed, if $\hat{\vartheta}^{-n_k}\rho \to \delta_x$ then $\hat{\vartheta}\hat{\vartheta}^{-n_k}\rho \to \delta_{\hat{\vartheta}x}$ because $\hat{\vartheta}$ is a homeomorphism. On the other hand, we have $\hat{\vartheta}\hat{\vartheta}^{-n_k}\rho = \hat{\vartheta}^{-n_k}\hat{\vartheta}\rho = \mu * \hat{\vartheta}^{-n_k}\rho$ by Proposition 3.1(a) and the fact that $\hat{\vartheta}$ and the G-action commute. So $\delta_{\hat{\vartheta}x} = \mu * \delta_x$. Hence, $gx = \hat{\vartheta}x$ for all $g \in \operatorname{supp}\mu = S_1$. Since $\hat{\vartheta}$ commutes with the G-action we conclude that $gx = x$ for every $g \in \bigcup_{n=1}^{\infty}(S_1^n S_1^{-n} \cup S_1^{-n}S_1^n)$. As N is the closed subgroup generated by $\bigcup_{n=1}^{\infty}(S_1^n S_1^{-n} \cup S_1^{-n}S_1^n)$ [6, Proposition 1.1] we obtain the desired result.

Note that $\mathcal{X}' = \{x \in \mathcal{X}; Nx = \{x\}\}$ is a closed subset of $\mathcal{X}$. As $\alpha(U) \neq 0$ for every nonempty open set and $\alpha(\mathcal{X} - \mathcal{X}') = 0$, we conclude that $\mathcal{X} = \mathcal{X}'$. Hence, it follows from Proposition 3.1(c) that N is contained in the group of periods of $\mathcal{H}^\infty$. If so, the proof is complete by Theorem 4.4(b). ∎

We recall that every recurrent random walk is irreducible. Hence, Theorem 1.4 follows from Theorem 5.1, Proposition 4.3, and Corollary 4.7. Theorem 1.5 follows from Corollary 4.6 and Theorem 5.1.

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Appendix: A proof of Proposition 3.1

LEMMA A.1: *Let G be a lcsc group and f an element of $L^\infty(G)$ such that $\lim_{g\to e}\|gf - f\| = 0$. Then f is left uniformly continuous (i.e., it is an equivalence class of a left uniformly continuous function).*

Proof: Let $\{V_n\}_{n=0}^{\infty}$ be a decreasing neighbourhood base at e and let $\epsilon_n \in L^1(G)$ be a probability measure carried on V_n. Define

$$f_n(x) = (\epsilon_n * f)(x) = \int \epsilon_n(dg) f(g^{-1}x).$$

Then f_n is a bounded left uniformly continuous function. Our assumption implies that for a given $\epsilon > 0$ there exists $N \geq 1$ such that $\|gf - f\| \leq \epsilon/2$ whenever $n \geq N$ and $g \in V_n$. Hence, for $n \geq N$ and $\alpha \in L^1(G)$, we have

$$|\langle \alpha, f_n - f\rangle| = \left| \int \alpha(dx) \int \epsilon_n(dg)(f(g^{-1}x) - f(x)) \right|$$
$$\leq \int |\alpha|(dx) \int \epsilon_n(dg)|f(g^{-1}x) - f(x)| \leq \|\alpha\|\epsilon/2.$$

Thus $\|f_n - f\| \leq \epsilon/2$, and we conclude that $\|f_n - f_m\| \leq \epsilon$ whenever $n, m \geq N$. Since the f_n's are bounded left uniformly continuous functions, this means that the sequence f_n converges uniformly to a bounded left uniformly continuous function f_*. On the other hand, f_n converges to f in $L^\infty(G)$. Hence, f is left uniformly continuous as claimed. ∎

Proof of Proposition 3.1: We remark that for a given $f \in C_0(\mathcal{X})$ the map $G \ni g \to gf$ is continuous with respect to the sup norm on $C_0(\mathcal{X})$. Since this norm majorizes the $L^\infty(\mathcal{X}, \mathcal{A}, \alpha)$ norm and the maps $R\theta^{-n}$: $L^\infty_a(\mu) \to L^\infty(G)$ are contractions, Lemma A.1 implies that for every n, $R\theta^{-n}\Phi f \in L^\infty(G)$ can be identified with a unique continuous function. Hence, we can define a linear functional $\hat{\rho}_n$ on $C_0(\mathcal{X})$ by setting $\hat{\rho}_n f = (R\theta^{-n}\Phi f)(e)$. Clearly, $\|\hat{\rho}_n\| \leq 1$. By the Riesz representation theorem there is then a unique finite measure ρ_n on $\mathcal{X}$ such that $\hat{\rho}_n f = \int \rho_n(dx)f(x)$ for all $f \in C_0(\mathcal{X})$. Using the equivariance of $R\theta^{-n}$ and Φ it is easy to see that $(R\theta^{-n}\Phi f)(g) = \int \rho_n(dx)f(gx)$ for all $f \in C_0(\mathcal{X})$ and all $g \in G$. Since for every $f \in C_0(\mathcal{X})$, $\{R\theta^{-n}\Phi f\}_{n=0}^\infty \in \mathcal{H}^\infty$ and all $R\theta^{-n}\Phi f$'s are continuous, from the uniqueness of ρ_n we conclude that $\mu * \rho_{n+1} = \rho_n$ for all $n = 0, 1, \ldots$. Let $\hat{\theta}$: $L^\infty(\mathcal{X}, \mathcal{A}, \alpha) \to L^\infty(\mathcal{X}, \mathcal{A}, \alpha)$ be the automorphism $\hat{\theta} f = f \circ \hat{\vartheta}$. Since, $R\theta^{-n-1}\Phi\hat{\theta} = R\theta^{-n}\Phi$, again by the uniqueness of ρ_n we obtain $\hat{\vartheta}\rho_{n+1} = \rho_n$. We set $\rho = \rho_0$. Then $\mu * (\hat{\vartheta}^{-1}\rho) = \mu * \rho_1 = \rho_0 = \rho$ and (a) follows since $\hat{\vartheta}$ commutes with the G-action.

To prove (b) consider the dual map $(R\Phi)^*$: $L^1(G) \to L^1(\mathcal{X}, \mathcal{A}, \alpha)$. As a direct consequence of Φ being an isomorphism we have that $\Phi^* Q_\nu \sim \alpha$, where ν is a finite measure equivalent to λ. On the other hand, $Q_\nu = R^*\nu$. Hence, $\alpha \sim (R\Phi)^*\nu$. But if $f \in C_0(\mathcal{X})$ then

$$\langle (R\Phi)^*\nu, f\rangle = \langle \nu, R\Phi f\rangle = \int \nu(dg) \int \rho(dx) f(gx) = \int (\nu * \rho)(dx) f(x).$$

So by the Riesz representation theorem $(R\Phi)^*\nu = \nu * \rho$, and thus $\alpha \sim \nu * \rho \sim \lambda * \rho$. Since $\hat{\vartheta}\alpha \sim \alpha$, (b) follows.

Having established (b) we can define weakly$*$ continuous equivariant contractions R'_n: $L^\infty(\mathcal{X},\mathcal{A},\alpha) \to L^\infty(G)$ by setting $(R'_n f)(g) = \int(\hat{\vartheta}^{-n}\rho)(dx)f(gx)$. Clearly, $R'_n f = R\theta^{-n}\Phi f$ whenever $f \in C_0(\mathcal{X})$. But $C_0(\mathcal{X})$ is weakly$*$ dense in $L^\infty(\mathcal{X},\mathcal{A},\alpha)$ (an easy consequence of Lusin's theorem [20, Theorem 2.23]). Hence, (c) is true. Since $R\Phi\mathbb{1} = \mathbb{1}$ we also conclude that ρ is a probability measure.

(d) follows from Theorem 2.3 and the equalities $\|Q_\varphi\|_a = \|R^*\varphi\|_a = \|\Phi^* R^*\varphi\| = \|(R\Phi)^*\varphi\| = \|\varphi * \rho\|$.

We now proceed to the proof of the convergence to point measure. Note first that for a given $f \in C_0(\mathcal{X})$ the sequence

$$\int(\omega_n\hat{\vartheta}^{-n}\rho)(dx)f(x) = \int(\hat{\vartheta}^{-n}\rho)(dx)f(\omega_n x)$$

is convergent for Q_λ-a.e. $\omega = \{\omega_n\}_{n=0}^\infty \in G^\infty$ because $\{\int(\hat{\vartheta}^{-n}\rho)(dx)f(\cdot x)\}_{n=0}^\infty$ is a space-time harmonic function (Theorem 2.1). Weak$*$ convergence of a sequence $\{\nu_n\}_{n=1}^\infty$ of probability measures is equivalent to convergence of the sequence $\int \nu_n(dx)f(x)$ for every f belonging to a dense subset of $C_0(\mathcal{X})$. But $C_0(\mathcal{X})$ is separable because $\mathcal{X}$ is second countable. Hence,

$$\Omega' = \{\omega \in G^\infty;\ \text{the sequence } \omega_n\hat{\vartheta}^{-n}\rho \text{ converges weakly}*\ \} \in \mathcal{B}^{(a)}$$

and $Q_\lambda(G^\infty - \Omega') = 0$. Moreover, if $\omega \in \Omega'$ and $\rho_\omega = \lim_{n\to\infty}\omega_n\hat{\vartheta}^{-n}\rho$, then by Theorem 2.1 we have $\int \rho_\omega(dx)f(x) = (\Phi f)(\omega)$ Q_λ-a.e. for every $f \in C_0(\mathcal{X})$. Since Φ is multiplicative, we can conclude that

$$\int \rho_\omega(dx)f_1(x)f_2(x) = \Big(\int \rho_\omega(dx)f_1(x)\Big)\Big(\int \rho_\omega(dx)f_2(x)\Big)$$

Q_λ-a.e. for every pair $f_1, f_2 \in C_0(\mathcal{X})$. Using again the separability of $C_0(\mathcal{X})$ we obtain that

$$\Omega'' = \Big\{\omega \in \Omega';\ \text{the functional } C_0(\mathcal{X}) \ni f \to \int \rho_\omega(dx)f(x) \text{ is multiplicative}\Big\} \in \mathcal{B}^{(a)}$$

and $Q_\lambda(G^\infty - \Omega'') = 0$. Thus for $\omega \in \Omega''$, ρ_ω is either zero or a point measure. Let $A = \{\omega \in \Omega''; \rho_\omega = 0\}$. By the separability of $C_0(\mathcal{X})$, $A \in \mathcal{B}^{(a)}$. We want to show that $Q_\lambda(A) = 0$. Let $\beta \in L^1_a(\mu)$ be defined by $d\beta/dQ_\nu = \chi_A$ where ν is a finite measure equivalent to λ. Since for every $f \in C_0(\mathcal{X})$, $(\Phi f)(\omega) = \int \rho_\omega(dx)f(x)$

Q_λ-a.e., we obtain that $\langle \beta, \Phi f\rangle = 0$ for all $f \in C_0(\mathcal{X})$. Using the density of $C_0(\mathcal{X})$ in $L^\infty(\mathcal{X}, \mathcal{A}, \alpha)$ and the fact that Φ is an isomorphism, we conclude that $\beta = 0$, i.e., that $Q_\nu(A) = 0$. Thus $Q_\lambda(A) = 0$ and the set

$$\begin{aligned}\Omega &= \{\omega \in G^\infty;\ \text{the sequence } \omega_n \hat{\vartheta}^{-n}\rho \text{ converges weakly* to a point measure}\}\\ &= \Omega'' - A\end{aligned}$$

has the properties asserted in Proposition 3.1.

Define now φ: $\Omega \to \mathcal{X}$ by $\delta_{\varphi(\omega)} = \lim_{n\to\infty} \omega_n \hat{\vartheta}^{-n}\rho = \rho_\omega$. It follows that for every $f \in C_0(\mathcal{X})$, $f \circ \varphi$ is a $\mathcal{B}^{(a)}$-measurable function defined on Ω. But from the basic properties of lcsc Hausdorff spaces we have that for every open $U \subseteq \mathcal{X}$ there is a sequence $f_n \in C_0(\mathcal{X})$ converging pointwise to χ_U. Hence, φ is a $\mathcal{B}^{(a)}$-measurable map of Ω into $\mathcal{X}$. Let F be any $\mathcal{B}^{(a)}$-measurable extension of φ onto G^∞. Consider the measure FQ_ν on $\mathcal{X}$. Let $f \in C_0(\mathcal{X})$. Since $(\Phi f)(\omega) = f(F(\omega))$ Q_ν-a.e., we obtain $\int (FQ_\nu)(dx) f(x) = \int Q_\nu(d\omega) f(F(\omega)) = \langle Q_\nu, \Phi f\rangle = \langle \Phi^* Q_\nu, f\rangle$. Hence, $FQ_\nu = \Phi^* Q_\nu \sim \alpha$ and we can define a weakly* continuous $*$-homomorphism Φ': $L^\infty(\mathcal{X}, \mathcal{A}, \alpha) \to L^\infty_a(\mu)$ by $\Phi' f = f \circ F$. Clearly, $\Phi' f = \Phi f$ for all $f \in C_0(\mathcal{X})$. By the density of $C_0(\mathcal{X})$ in $L^\infty(\mathcal{X}, \mathcal{A}, \alpha)$, $\Phi' = \Phi$.

It remains to prove the last statement of the Proposition. Note that $\Omega_2 = \Omega \cap \Omega_1$ is a Borel subset of the standard Borel space $(G^\infty, \mathcal{B}^\infty)$. Hence, by [1, Theorem 3.3.4], $F(\Omega_2)$ is an analytic set in $\mathcal{X}$. If so, then by [1, Theorem 3.2.4] there are Borel sets $\mathcal{X}_1, \mathcal{X}_2 \subseteq \mathcal{X}$ such that $\mathcal{X}_1 \subseteq F(\Omega_2) \subseteq \mathcal{X}_2$ and $\alpha(\mathcal{X}_2 - \mathcal{X}_1) = 0$. Our proof will be complete if we show that $\alpha(\mathcal{X} - \mathcal{X}_2) = 0$. But this follows directly from the fact that Φ is injective. ∎

References

[1] W. Arveson, *Invitation to C^*-algebras*, Springer, New York, 1976.

[2] R. Azencott, *Espaces de Poisson des groupes localement compacts*, Lecture Notes in Mathematics, Vol. 148, Springer, Berlin, 1970.

[3] Y. Derriennic, *Lois zero ou deux pour les processus de Markov. Applications aux marches aleatories*, Annales de l'Institut Henri Poincaré **12** (1976), 111–129.

[4] Y. Derriennic, *Entropie, theoremes limites et marches aleatoires*, in *Probability Measures on Groups* (H. Heyer, ed.), Proceedings, Oberwolfach 1985, Lecture Notes in Mathematics, Vol. 1210, Springer, Berlin, 1986, pp. 241–284.

[5] Y. Derriennic and M. Lin, *Sur la tribu asymptotique des marches aleatoires sur les groupes,* Publications des Seminaires de Mathématiques, Institut de Recherche Mathématiques de Rennes, 1983.

[6] Y. Derriennic and M. Lin, *Convergence of iterates of averages of certain operator representations and of convolution powers,* Journal of Functional Analysis **85** (1989), 86–102.

[7] Y. Derriennic and M. Lin, *Sur le comportement asymptotique des puissances de convolution d'une probabilité,* Annales de l'Institut Henri Poincaré, Probabilités **20** (1984), 127–132.

[8] S. R. Foguel, *On iterates of convolutions,* Proceeding of the American Mathematical Society **47** (1975), 368–370.

[9] H. Furstenberg, *Random walks and discrete subgroups of Lie groups,* Advances in Probability and Related Topics **1**, Marcel Dekker, New York, 1971, pp. 3–63.

[10] H. Furstenberg, *Boundary theory and stochastic processes on homogeneous spaces,* Proceedings of Symposia in Pure Mathematics, Vol. 26: Harmonic analysis on homogeneous spaces, American Mathematical Society, Providence, R.I., 1973, pp. 193–229.

[11] F. P. Greenleaf, *Invariant Means on Topological Groups and Their Applications,* Van Nostrand, New York, 1969.

[12] E. Hewitt and L. J. Savage, *Symmetric measures on Cartesian products,* Transactions of the American Mathematical Society **80** (1955), 470–501.

[13] B. Jamison and S. Orey, *Markov chains recurrent in the sense of Harris,* Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete **8** (1967), 41–48.

[14] M. Lin and R. Wittmann, *Convergence of representation averages and convolution powers,* Israel Journal of Mathematics **88** (1994), 125–157.

[15] G. W. Mackey, *Point realizations of transformation groups,* Illinois Journal of Mathematics **6** (1962), 327–335.

[16] P. A. Meyer, *Probability and Potentials,* Blaisdell, Waltham, 1966.

[17] J. Neveu, *Mathematical Foundations of the Calculus of Probability,* Holden-Day, San Francisco, 1965.

[18] A. L. T. Paterson, *Amenability,* American Mathematical Society, Providence, R. I., 1988.

[19] D. Revuz, *Markov Chains,* North-Holland, Amsterdam, 1984.

[20] W. Rudin, *Real and Complex Analysis,* McGraw-Hill, New York, 1974.

[21] V. A. Varadarajan, *Groups of automorphisms of Borel spaces,* Transactions of the American Mathematical Society **109** (1963), 191–220.